

# Deformed Ginibre ensembles and integrable systems

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## Abstract

We consider three Ginibre ensembles (real, complex and quaternion-real) with a deformed measure and relate them to the known integrable systems via presenting partition functions of these ensembles in form of fermionic expectation values. We also introduce double deformed Dyson-Wigner ensembles and compare their fermionic representations with these of Ginibre ensembles.

**Key words:** integrable systems, tau functions, Pfaffians, BKP, DKP, two-component Toda lattice, free fermions, double deformed Dyson-Wigner ensembles, double deformed Ginibre ensembles.

## 1 Introduction

Ginibre ensembles play an important role in many statistical problems. They were introduced in 1965 by Ginibre as non Hermitian analogues of famous Wigner-Dyson ensembles, namely, ensembles of random real symmetric, Hermitian and quaternion self-dual random matrices, known also as orthogonal, unitary and symplectic Wigner-Dyson ensembles. The non-Hermitian analogues are called real Ginibre, complex Ginibre and quaternion-real Ginibre ensemble respectively.

Let us recall that both Wigner-Dyson ensembles and Ginibre counterparts are Gauss ones. This was enough for the problems considered by physicists working in quantum chaos. It is widely known that the deformation of the Gauss measure of each Wigner-Dyson ensemble makes them tau-functions of integrable hierarchies where deformation parameters treated as the so-called higher times. First this link was established for the unitary ensemble which was identified with the 1D Toda lattice tau function [12], and later also for the orthogonal and symplectic ones which were related to the so-called Pfaff lattice [7].

The origin for the deformation of WD ensembles were applications of these models to certain problems in physics (string theory and 2D quantum gravity and other application of summation over poly-angulated surfaces) and in mathematics (few enumeration problems). At present there are no special motivation to consider deformations of the Ginibre ensembles, however one may believe in future applications.

Wigner and Dyson had physical reasons to consider orthogonal, unitary and symplectic ensembles. Physical systems are described by Hamiltonian operators which are Hermitian. Ginibre had no physical reasons and introduced non-Hermitian analogues from "mathematical" point of view, having a hope that they may be of use in future. Indeed at present Ginibre ensembles are in a focus of attention in many fields (such as quantum chromodynamics, dissipative quantum maps, scattering in chaotic quantum systems, growth processes, fractional quantum-Hall effect, Coulomb plasma, stability of complex biological and neural networks, directed quantum chaos in randomly pinned superconducting vortices, delayed time series in financial markets, random operations in quantum information theory, see a review article [1] for details.)

I am going to show that the partition function of the real and quaternion-real Ginibre ensembles are related to integrable systems similar to many other matrix models [12], [13], [21], [23], [7], [5] which may be referred as deformed Dyson-Wigner ensembles. Here we use the so-called "large" BKP hierarchy introduced in [4] (named by authors "charged" or "fermionic" BKP) and the "large" 2-BKP hierarchy introduced in [3]. The last hierarchy is rather similar to 2-DKP (Pfaff-DKP) one introduced in [20]<sup>1</sup> We

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<sup>1</sup> Large BKP hierarchy includes the famous KP one as a particular reduction, while large 2-BKP includes the Toda lattice hierarchy [19]. In the present text we will refer these hierarchies as BKP and 2-BKP ones instead of the "large BKP" and "large 2-BKP". Make difference between these "large" BKP hierarchy containing KP and the "small" (referred in [4] as "neutral") one introduced in [28] and which is a subhierarchy of the KP hierarchy.

shall see that real and quaternion-real Ginibre ensemble both deformed in a appropriate way may be related to these hierarchies where the higher times, namely, an integer  $L$  and a pair of semi-infinite sets of numbers  $\mathbf{t} = (t_1, t_2, \dots)$  and  $\mathbf{s} = (s_1, s_2, \dots)$  plays the role of deformation parameters. We say that this is “double” deformed ensembles because two sets of parameters is used.

To be more precise it is only the case where  $L \geq 0$  and  $\mathbf{s} = 0$  may be related to what is called Ginibre ensembles. To consider more general deformations we need to introduce *restricted* (real, quaternion-real, complex) Ginibre ensembles where the space of random matrices consists of only invertable (real, quaternion-real, complex) matrices. Then we prove that partition functions of deformed (real and quaternion-real) Ginibre ensembles are BKP tau functions, where  $\mathbf{t}$  and  $L \geq 0$  are deformation parameters and BKP higher times. While partition functions of double deformed restricted (real and quaternion-real) ensembles are 2-BKP tau functions, where now the set  $L, \mathbf{t}, \mathbf{s}$  are deformation parameters and higher times.

A fermionic representation for the tau functions will be written down and compared with the representation of appropriately deformed classical Dyson-Wigner ensembles, where an additional set of parameters  $\mathbf{s}$  is added. The complex Ginibre ensemble will be deformed with the help of four sets of parameters,  $\mathbf{t}, \mathbf{t}', \mathbf{s}, \mathbf{s}'$ , and related to the two-component Toda lattice.

## 2 Gauss Real Ginibre ensemble and its deformation

**Real Hermitian ensemble.** First let us remind that the ensemble of real Hermitian matrices (also known as Dyson-Wigner orthogonal ensemble) is given by the following partition function

$$I_N^{d-OE}(\mathbf{t}) = \int d\mu(X, \mathbf{t}), \quad d\mu(X, \mathbf{t}) = e^{-\frac{1}{2}\text{tr}(X^2) + \text{tr}V(X, \mathbf{t})} \prod_{i \geq j} dX_{ij} \quad (1)$$

$$V(x, \mathbf{t}) := \sum_{n=1}^{\infty} x^n t_n \quad (2)$$

where  $X$  is  $N$  by  $N$  real Hermitian matrix and  $\mathbf{t} = (t_1, t_2, \dots)$  is a set of parameters which describe the deviation of the probability measure from the Gauss one.

Let us restrict the space of our symmetric matrices to invertable symmetric matrices. Then more general deformation of the measure may be considered as follows

$$I_N^{dd-OE}(L, \mathbf{t}, \mathbf{s}) = \int d\mu(X, L, \mathbf{t}, \mathbf{s}), \quad d\mu(X, L, \mathbf{t}, \mathbf{s}) = \det X^L e^{-\text{tr}V(X^{-1}, \mathbf{s})} d\mu(X, \mathbf{t}) \quad (3)$$

where  $\mathbf{s}$  and an integer  $L$  is the collection of new deformation parameters. Dyson-Wigner Gauss ensemble will be referred as G-OE, ensembles (1) and (3) respectively as d-OE and dd-OE.

It was found by M.Adler and P. van Moerbeke in [7] that the ensemble (1) may be related to the so-called Pfaff Toda lattice [8]. Later in [5] J. van de Leur found that the orthogonal ensemble (1) may be also identified with a tau function of the “large” BKP tau function introduced in [4]. Below we shall identify the ensemble (3) with a “large” 2-BKP tau function introduced in [3].

In terms of eigenvalues  $x_1, \dots, x_N$  of the Hermitian matrix  $X$  the integral (3) may be written as

$$I_N^{dd-OE}(L, \mathbf{t}, \mathbf{s}) = a_N \int_{x_1 > \dots > x_N} \Delta(x_1, \dots, x_N) \prod_{i=1}^N x_i^L e^{-\frac{1}{2}x_i^2 + V(x_i, \mathbf{t}) - V(x_i^{-1}, \mathbf{s})} dx_i \quad (4)$$

with some constant  $a_N$  related to the volume of the orthogonal group  $O(N)$ . Deformation parameters  $\mathbf{t}$  and  $\mathbf{s}$  are considered to be chosen in a way that the integral (4) is convergent.

**Real non-Hermitian ensemble.** Let us turn to the so-called Ginibre ensembles. Gauss real Ginibre ensemble, also known as Gauss Ginibre orthogonal ensemble (G-GinOE) is defined on the space of real matrices by assigning Gauss probability measure to each entry with the same variance:

$$I_N^{G-GinOE} = \int d\mu(X), \quad d\mu(X) = \prod_{i,j} e^{-\frac{1}{2}X_{ij}^2} dX_{ij} = e^{-\frac{1}{2}\text{tr}(XX^\dagger)} \prod_{i,j} dX_{ij} \quad (5)$$

where  $X$  is  $N$  by  $N$  matrix with real entries. Measure (5) is invariant under the orthogonal transformations of matrices  $X$ .

The so-called elliptic deformation of G-GinOE measure as

$$d\mu(X, a) = e^{-a \text{tr} X^2} d\mu(X) \quad (6)$$

where  $a$  is the deformation parameter was studied in [1]<sup>2</sup>.

We shall consider the following deformation of the G-GinOE ensemble which we will refer as d-GinOE and which includes the elliptic deformation as a particular case

$$I_N^{d-GinOE}(L, \mathbf{t}) = \int d\mu(X, L, \mathbf{t}), \quad d\mu(X, L, \mathbf{t}) = \det X^L e^{\text{tr} V(X, \mathbf{t})} d\mu(X) \quad (7)$$

where  $V$  is given by (2) and where  $L = 0, 1, 2, \dots$  and sets of numbers  $\mathbf{t} = (t_1, t_2, \dots)$  are deformation parameters. We assume that  $\mathbf{t}$  are chosen in a way that the partition function  $I_N^{d-GinOE}(L, \mathbf{t})$  is finite.

*Remark 1.* However even in the case where the integral (7) is divergent itself, logarithmic derivatives with respect to the deformation parameters may be finite. How it works in the models of normal matrices, see in [6].

Now let us consider the restricted Ginibre ensemble consisting of real invertable matrices and deform its measure as follows

$$I_N^{dd-GinOE}(L, \mathbf{t}, \mathbf{s}) = \int d\mu(X, L, \mathbf{t}, \mathbf{s}), \quad d\mu(X, L, \mathbf{t}, \mathbf{s}) = \det X^L e^{\text{tr} V(X, \mathbf{t}) - \text{tr} V(X^{-1}, \mathbf{s})} d\mu(X) \quad (8)$$

where  $X$  are real invertable  $N$  by  $N$  matrices and where  $V$  is given by (2) and where an integer  $L$  and sets of numbers  $\mathbf{t} = (t_1, t_2, \dots)$ ,  $\mathbf{s} = (s_1, s_2, \dots)$  are deformation parameters. This ensemble will be called double deformed restricted Ginibre orthogonal ensemble (dd-GinOE). Again we suppose that deformation parameters are chosen in a way which provides the existence of the integral  $I_N^{dd-GinOE}(L, \mathbf{t}, \mathbf{s})$ . Consider the following example. If one takes  $\mathbf{t} \rightarrow \mathbf{t} - \sum_{i=1}^{N_1} a_i [p_i]$  and  $\mathbf{s} \rightarrow \mathbf{s} - \sum_{i=1}^{N_1} b_i [q_i]$ , namely

$$t_n \rightarrow t_n - \frac{1}{n} \sum_{i=1}^{N_1} a_i p_i^n, \quad s_n \rightarrow s_n - \frac{1}{n} \sum_{j=1}^{N_2} b_j q_j^n \quad (9)$$

then the integral (8) reads as

$$\int \prod_{i=1}^{N_1} \det(1 - p_i X)^{a_i} \prod_{j=1}^{N_2} \det(1 - q_j X^{-1})^{-b_j} d\mu(X, L, \mathbf{t}, \mathbf{s}) \quad (10)$$

Let prove that the integrals (7) and (8) may be identified with tau functions of the "large" BKP hierarchy and 2-BKP respectively. Then the size of matrices  $N$  and the set  $L, \mathbf{t}, \mathbf{s}$  play the role of the so-called higher times of this integrable hierarchy.

To do it we should re-write integral (8) as the integral over eigenvalues. Let us remind that for the Gauss real Ginibre ensemble (namely, for the case  $\mathbf{t} = \mathbf{s} = 0$ ,  $L = 0$ ) this problem was solved in [2], see also a review article [1]. In very short it was done as follows. After applying the Schur decomposition  $X = U(\Lambda + \Delta)U^\dagger$ , where  $U$  is orthogonal,  $\Lambda$  is block-diagonal and  $\Delta$  has nonzero blocks only above  $\Lambda$  they arrive at

$$d\mu(X) = e^{-2\text{Tr}(\Delta\Delta^\dagger + \Lambda\Lambda^\dagger)} |D\Delta| |D\Lambda| \prod' (U^{-1}dU)_{ij} \frac{(\lambda_i - \lambda_j)}{2\pi}$$

with the dashed product running over non-zero entries in the lower triangle of  $U^\dagger dU$ . The set of the eigenvalues  $\lambda_i$ ,  $i = 1, \dots, N$  of a real matrix  $X$  consists of real numbers and of complex conjugated pairs, denoted below as  $x_i$  and  $(z_i, \bar{z}_i)$  respectively. Thus, the matrix  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$  may be written as  $\text{diag}(z_1, \bar{z}_1, \dots, z_k, \bar{z}_k, x_1, \dots, x_{N-2k})$  with certain  $k$ . After integration over matrices  $\Delta$  and  $U$  they come to the integral over eigenvalues only which includes integral over real values  $x_i$  and over complex

<sup>2</sup> It is also called the elliptic deformation of G-GinOE because in the large  $N$  limit the parameter  $a$  describes a deformation of a circle equilibrium domain of the eigenvalues in complex plane to an elliptic form.

eigenvalues  $z_j, \bar{z}_j$ . Finely, after certain computations [2] the Gauss real Ginibre ensemble may be written as

$$I_N^{G-GinOE} = b_N \cdot \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \int_{\mathbb{M}_{2k, N-2k}} \Delta_{2n}(z_1, \bar{z}_1, \dots, z_k, \bar{z}_k, x_1, \dots, x_{N-2k}) d\Omega_{2k}^C d\Omega_{N-2k}^R \quad (11)$$

where the integration domain  $\mathbb{M}_{2m, N-2m}$  is as follows  $\Re z_1 > \dots > \Re z_m, x_{2m+1} > \dots > x_N, z_i \in \mathbb{C}_+$  (upper half-plane),  $x_i \in \mathbb{R}$ , and where

$$d\Omega_{2m}^C(\mathbf{z}, \bar{\mathbf{z}}) = \prod_{i=1}^m \operatorname{erfc}\left(\frac{|z_i - \bar{z}_i|}{\sqrt{2}}\right) e^{-\Re z_i^2} d^2 z_i, \quad d\Omega_{N-2m}^R(\mathbf{x}) = \prod_{i=2m+1}^N e^{-\frac{1}{2}x_i^2} dx_i \quad (12)$$

with  $\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-x^2} dx$ . Here the factor  $c_N$  absorbs integrals over  $\Delta$  and over  $U$ . Let us note that the Gauss case is a special case of (8):  $I_N^{G-GinOE} = I_N^{dd-GinOE}(0, 0, 0)$ . The factor  $b_N$  is independent of  $L, \mathbf{t}, \mathbf{s}$  and was evaluated for GinOE in [2].

Now we notice that the deformation (8) results in the multiplication of the measure  $d\mu(X)$  by a factor which depends only on eigenvalues of the matrix  $X$

$$d\mu(X) \rightarrow d\mu(X, \mathbf{t}, L, \mathbf{s}) = d\mu(X) \prod_{i=1}^N \lambda_i^L e^{V(\lambda_i, \mathbf{t}) - V(\lambda_i^{-1}, \mathbf{s})} \quad (13)$$

where  $\lambda_i$  are eigenvalues of  $X$  and  $V$  is given by (2). Then we introduce

$$d\Omega_{2m}^C \rightarrow d\Omega_{2m}^C(\mathbf{t}, L, \mathbf{s}) = \prod_{i=1}^m \operatorname{erfc}\left(\frac{|z_i - \bar{z}_i|}{\sqrt{2}}\right) |z_i|^{2L} e^{2\Re V(z_i, \mathbf{t}) - 2\Re V(\bar{z}_i^{-1}, \mathbf{s}) - \Re z_i^2} d^2 z_i \quad (14)$$

$$d\Omega_{N-2m}^R \rightarrow d\Omega_{N-2m}^R(\mathbf{t}, L, \mathbf{s}) = \prod_{i=2m+1}^N x_i^L e^{V(x_i, \mathbf{t}) - V(x_i^{-1}, \mathbf{s}) - \frac{1}{2}x_i^2} dx_i \quad (15)$$

Let us introduce

$$J_N(L, \mathbf{t}, \mathbf{s}; \alpha) = \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \alpha^k \int_{\mathbb{M}_{2k, N-2k}} |z_i|^{2L} \Delta_{2n}(z_1, \bar{z}_1, \dots, z_k, \bar{z}_k, x_1, \dots, x_{N-2k}) d\Omega_{2k}^C(\mathbf{t}, L, \mathbf{s}) d\Omega_{N-2k}^R(\mathbf{t}, L, \mathbf{s}) \quad (16)$$

We note that for  $\alpha \rightarrow 0$  (up to a constant independent of the variables  $L, \mathbf{t}, \mathbf{s}$ ) this expression is equal to the partition function of the deformed Dyson-Wigner orthogonal ensemble (7). When  $\alpha = 1$  this is the partition function for the deformed real Ginibre ensemble (33).

$$J_N(L, \mathbf{t}, \mathbf{s} = 0; \alpha = 1) = b_N I_N^{d-GinOE}(L, \mathbf{t}) \quad (17)$$

$$J_N(L, \mathbf{t}, \mathbf{s}; \alpha = 0) = a_N I_N^{dd-OE}(L, \mathbf{t}, \mathbf{s}), \quad J_N(L, \mathbf{t}, \mathbf{s}; \alpha = 1) = b_N I_N^{dd-GinOE}(L, \mathbf{t}, \mathbf{s}) \quad (18)$$

Independent of deformation parameters factors  $a_N$  and  $b_N$  may be found respectively in [10] and [1].

**Tau functions and matrix ensembles.** Let us remind the fermionic construction of the so-called tau functions [9]. Namely here we present a general fermionic expression for the 2-BKP tau function and relate it to ensembles (3) and (8). In the next section it will be also related to a deformed symplectic Dyson-Wigner ensemble and to a deformed quaternion-real Ginibre ensemble.

The very notion of tau function and it's fermionic construction was introduced by Kyoto school. Here we use it in a version suggested in [4] where an additional fermionic mode  $\phi$  was added. see Appendix A.2.

Following [9] we consider

$$\Gamma_+(\mathbf{t}) = \exp \sum_{n=1}^{\infty} t_n \sum_{i \in \mathbb{Z}} \psi_i \psi_{i+n}^\dagger, \quad \Gamma_-(\mathbf{s}) = \exp \sum_{n=1}^{\infty} s_n \sum_{i \in \mathbb{Z}} \psi_i \psi_{i-n}^\dagger \quad (19)$$

We shall need the following equality

$$\Gamma_+(\mathbf{t}) \psi(z) \Gamma_-(\mathbf{s}) = c(\mathbf{t}, \mathbf{s}) e^{V(z, \mathbf{t}) - V(z^{-1}, \mathbf{s})} \Gamma_-(\mathbf{s}) \psi(z) \Gamma_+(\mathbf{t}) \quad (20)$$

where

$$c(\mathbf{t}, \mathbf{s}) = \exp \sum_{n=1}^{\infty} n t_n s_n \quad (21)$$

The 2-BKP tau function may be presented in form of the following fermionic expectation value

$$\tau_N(L, \mathbf{t}, \mathbf{s}) = \langle N + L | \Gamma_+(\mathbf{t}) e^{\Phi} \Gamma_-(\mathbf{s}) | L \rangle \quad (22)$$

where  $\Phi$  is any quadratic expression in fermionic modes  $\psi_i, \psi_i^\dagger, \phi$ , see Appendix A.2. Variables  $N, L, \mathbf{t}, \mathbf{s}$  plays the role of the 2-BKP higher times. Tau function (22) solves 2-BKP Hirota equations, see [3] and Appendix A.6.

*Remark 2.* The 2-BKP tau function is also a certain BKP tau function [4] with respect to the variables  $N, L, \mathbf{t}$  and a certain BKP tau function with respect to the variables  $N, L, \mathbf{s}$  (this explains the name 2-BKP).

Tau function (22) contains as a particular cases: tau functions of the Toda lattice hierarchy [9], [19], the charged BKP tau function [4] and 2-DKP (“Pfaff DKP”) tau function [20]. Each of these cases may be obtained by specifying the tau function (22), namely, by specifying  $\Phi$  and  $\mathbf{s}$ , see Appendix A.4.

**Theorem 1.** *We have*

$$(-)^{NL} c(\mathbf{t}, \mathbf{s}) J_N(L, \mathbf{t}, \mathbf{s}; \alpha) = \langle N + L | \Gamma_+(\mathbf{t}) e^{\alpha \Phi_c} e^{\Phi_r} \Gamma_-(\mathbf{s}) | L \rangle \quad (23)$$

where

$$\Phi_c = \int_{\mathbb{C}_+} \operatorname{erfc} \left( \frac{|z - \bar{z}|}{\sqrt{2}} \right) \psi(z) \psi(\bar{z}) e^{-\Re z^2} d^2 z \quad (24)$$

is the integral over the upper half of complex plane and where

$$\Phi_r = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \operatorname{sgn}(x_1 - x_2) \psi(x_1) \psi(x_2) e^{-\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2} dx_1 dx_2 + \sqrt{2} \int_{\mathbb{R}} \psi(x) \phi e^{-\frac{1}{2}x^2} dx \quad (25)$$

Therefore thanks to (17) and Remark 2  $(-)^{NL} I_N^{d-GinOE}(L, \mathbf{t})$  is a tau function of the “large” BKP while thanks to (18) both  $(-)^{NL} c(\mathbf{t}, \mathbf{s}) I_N^{dd-OE}(L, \mathbf{t}, \mathbf{s})$  and  $(-)^{NL} c(\mathbf{t}, \mathbf{s}) I_N^{dd-GinOE}(L, \mathbf{t}, \mathbf{s})$  of the “large” 2-BKP ones.

The sketch of the proof of (23). We transform the vacuum expectation value in the right hand side to get the integral in the left hand side as follows. First we use Taylor expansion of the exponentials  $e^{\alpha \Phi_c}$  and  $e^{\beta \Phi_r}$  where only terms of the order  $\alpha^{N-2k} \beta^k$ ,  $k = 0, \dots, \lfloor \frac{N}{2} \rfloor$  are non-vanishing between  $\langle N + L |$  and  $|L \rangle$ . The term  $e^{\alpha \Phi_r}$  should be considered in a way similar to [5] (see Appendix). Notice that the Taylor expansion of the exponentials of integrals yields the integration domain  $\mathbb{M}_{2k, N-2k}$  when we re-write product of pairwise integrals as a  $N$ -fold integral. Next we send  $\Gamma$  to the right and  $\Gamma^*$  to the left taking into account relations  $\langle N + L | \Gamma^*(\mathbf{s}) = \langle N + L |$  and  $\Gamma(\mathbf{t}) | L \rangle = |L \rangle$  and using (20). In this way we get rid of operators  $\Gamma$  and  $\Gamma^*$  and instead obtain factors  $e^V$  inside integrals, the integrals are still between  $\langle N + L |$  and  $|L \rangle$ . At last we get rid of fermions thanks to

$$\langle N + L | \psi(z_1) \cdots \psi(z_N) | L \rangle = z_1^L \cdots z_N^L \Delta_N(\mathbf{z}), \quad \Delta_N(\mathbf{z}) := \prod_{i < j \leq N} (z_i - z_j) \quad (26)$$

obtained by a simple direct calculation. We obtain (23).

Now the fermionic expectation value  $\langle N + L | \Gamma(\mathbf{t}) e^{\alpha \Phi_c} e^{\Phi_r} \Gamma^*(\mathbf{s}) | L \rangle$  is an example of tau function with respect to the variables  $N, L, \mathbf{t}$  introduced in [4] where Hirota equations for such tau functions were written down. By symmetry it is also tau function with respect to the variables  $N, L, \mathbf{s}$ . Such (“coupled” BKP, or, the same 2-BKP) tau functions were considered in [3].

Thus we establish that the partition function of the deformed quaternion-real Ginibre ensemble (7) is the subject to the theory of integrable systems.

### 3 Quaternion-real Ginibre ensemble and its deformation

Each  $N \times N$  quaternion matrix (or, the same, a matrix with quaternion entries) may be viewed as  $2N \times 2N$  matrix if the quaternions  $e_n$ ,  $n = 1, 2, 3, 4$ , are realized as  $2 \times 2$  matrices:

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad (27)$$

Then quaternion-real matrix is a quaternion matrix where each entry is a linear combination of quaternions with real coefficients. Being viewed as  $2N \times 2N$  matrix a quaternion-real matrix  $X$  may be recognized via the property

$$EXE = -\bar{X} \quad (28)$$

where the bar means complex conjugate and where  $E$  is the block-diagonal  $2N \times 2N$  matrix with matrices  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  on the main diagonal; this follows from the explicit expression (27) and from  $e_2 e_j e_2 = e_j$ ,  $j = 1, 3$  and from  $e_2 e_j e_2 = -e_j$ ,  $j = 0, 2$ . Below we shall treat quaternion-real matrices as  $2N \times 2N$  matrices with the defining property (28).

**Quaternion-real Hermitian ensemble.** First we recall the case where in the quaternion-real matrix  $X$  is Hermitian (in this case  $X$  is called self-dual). The ensemble of random self-dual matrices is called symplectic ensemble, the Gauss symplectic ensemble (G-SE) is called Dyson-Wigner symplectic ensemble. We will consider a deformed symplectic ensemble (d-SE) as follows

$$I_N^{d-SE}(L, \mathbf{t}) = \int d\mu(X, L, \mathbf{t}), \quad d\mu(X, L, \mathbf{t}) = e^{-x^2 + 2\text{tr}V(X, \mathbf{t})} d\mu(X) \quad (29)$$

where  $d\mu(X)$  is the measure on the space of self-dual matrices (for details see [10]). As in the orthogonal ensemble case we may consider the subspace of invertable matrices and consider double deformed restricted ensemble of quaternion-real matrices (dd-SE)

$$I_N^{dd-SE}(L, \mathbf{t}, \mathbf{s}) = \int d\mu(X, L, \mathbf{t}, \mathbf{s}), \quad d\mu(X, L, \mathbf{t}, \mathbf{s}) = \det X^L e^{-x^2 + 2\text{tr}V(X, \mathbf{t}) - 2\text{tr}V(X^{-1}, \mathbf{s})} d\mu(X) \quad (30)$$

where we add the deformation parameters  $\mathbf{s} = (s_2, s_2, \dots)$  and integer  $L$ . Being re-written as an integral over eigenvalues of Hermitian quaternion real matrices  $X$  [10] it is as follows

$$I_N^{dd-SE}(L, \mathbf{t}, \mathbf{s}) = d_N \int (\Delta(x_1, x_2, \dots, x_N))^4 \prod_{i=1}^N x_i^{2L} e^{-x_i^2 + 2V(x_i, \mathbf{t}) - 2V(x_i^{-1}, \mathbf{s})} dx_i \quad (31)$$

where  $x_1, x_1, x_2, x_2, \dots, x_N, x_N$  are eigenvalues of the self-dual random  $2N$  by  $2N$  matrix  $X$  and the factor  $d_N$  does not depend on deformation parameters.

**Quaternion-real non-Hermitian ensemble.** Then Gauss quaternion-real Ginibre ensemble, also known as (Gauss) Ginibre symplectic ensemble (G-GinSE) is defined on the space of quaternion-real matrices by assigning the same Gauss probability measure to each entry:

$$I_N^{G-GinSE} = \int d\mu(X), \quad d\mu(X) = e^{-\frac{1}{2}\text{tr}(XX^\dagger)} \prod_{i,j} dX_{ij} \quad (32)$$

where  $X$  is treated as  $2N$  by  $2N$  matrix. Measure (32) is invariant under the symplectic transformations of matrices  $X$ .

*Remark 3.* A Hermitian quaternion-real matrix is called quaternion self-dual matrix. Ensemble (32) is non-Hermitian analogue of the ensemble of random quaternion self-dual matrices (known as Dyson-Wigner symplectic ensemble), see [10] for details.

The elliptic deformation of G-GinSE measure [1] is quite similar to (6):  $d\mu(X) \rightarrow e^{-a\text{tr}(X^2 + (X^\dagger)^2)} d\mu(X)$  where  $a$  is a deformation parameter.

The partition function for the deformed quaternionic-real Ginibre ensemble (d-GinSE) will be defined as

$$I_N^{d-GinSE}(L, \mathbf{t}) = \int d\mu(X, L, \mathbf{t}), \quad d\mu(X, L, \mathbf{t}) := \det X^L e^{\text{tr}V(X, \mathbf{t})} d\mu(X) \quad (33)$$

where a non-negative integer  $L$  and the set of numbers  $\mathbf{t} = (t_1, t_2, \dots)$  are deformation parameters.

On the space of invertable quaternion-real matrices we consider double deformed Ginibre ensemble

$$I_N^{dd-GinSE}(L, \mathbf{t}, \mathbf{s}) = \int d\mu(X, L, \mathbf{t}, \mathbf{s}), \quad d\mu(X, L, \mathbf{t}, \mathbf{s}) = \det X^L e^{\text{tr} V(X, \mathbf{t}) - \text{tr} V(X^{-1}, \mathbf{s})} d\mu(X) \quad (34)$$

where an integer  $L$  and the sets  $\mathbf{t} = (t_1, t_2, \dots)$ ,  $\mathbf{s} = (s_1, s_2, \dots)$  are deformation parameters. Here we assume that the integral is either convergent or regularized, see Remark 1.

Notice that if one takes

$$t_n \rightarrow t_n - \frac{1}{2n} \sum_{i=1}^n a_i p_i^n, \quad s_n \rightarrow s_n - \frac{1}{2n} \sum_{j=1}^n b_j q_j^n \quad (35)$$

then the integral (33) reads as

$$\int \det X^L \prod_i \det(1 - p_i X)^{a_i} \prod_j \det(1 - q_j X^{-1})^{-b_j} d\mu(X, L, \mathbf{t}, \mathbf{s}) \quad (36)$$

As in the case of real matrices all complex eigenvalues occur in complex conjugated pairs. Real eigenvalues have multiplicities no less than two. A calculation similar to calculation for real matrices yields [1] (see also [27], [24], [10])

$$I_N^{dd-GinSE}(L, \mathbf{t}, \mathbf{s}) = c_N \int_{\mathbb{M}_n} \Delta_{2n}(z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_n, \bar{z}_n) \prod_{i=1}^n |z_i - \bar{z}_i| |z_i|^{2L} e^{V(z_i, \mathbf{t}) + V(\bar{z}_i, \mathbf{t}) - V(z_i^{-1}, \mathbf{s}) - V(\bar{z}_i^{-1}, \mathbf{s}) - |z_i|^2} d^2 z_i \quad (37)$$

with some constant  $c_N$  where the integration domain  $\mathbb{M}_n$  consists of the sets of  $\mathbf{z}$  where  $\Re z_i > \Re z_{i+1}$  and  $\Im z_i > 0$ , and  $V$  is given by (2).

**Theorem 2.** *Introduce*

$$c(\mathbf{t}, \mathbf{s}) J_N(L, \mathbf{t}, \mathbf{s}, \alpha, \beta) = \langle 2N + L | \Gamma_+(\mathbf{t}) e^{\alpha \Phi_c} e^{\beta \Phi_r} \Gamma_-(\mathbf{s}) | N \rangle \quad (38)$$

where

$$\Phi_c^q = \int_{\mathbb{C}_+} (z - \bar{z}) \psi(z) \psi(\bar{z}) e^{-|z|^2} d^2 z \quad (39)$$

$$\Phi_r^q = \int_{\mathbb{R}} \frac{\partial \psi(x)}{\partial x} \psi(x) e^{-x^2} dx \quad (40)$$

is the integral over the upper half of complex plane. Then

$$I_N^{d-GinSE}(L, \mathbf{t}) = d_N J_N(L, \mathbf{t}, \mathbf{s} = 0, \alpha = 1, \beta = 0) \quad (41)$$

$$I_N^{dd-GinSE}(L, \mathbf{t}, \mathbf{s}) = d_N J_N(L, \mathbf{t}, \mathbf{s}, \alpha = 1, \beta = 0), \quad I_N^{dd-SE}(L, \mathbf{t}, \mathbf{s}) = c_N J_N(L, \mathbf{t}, \mathbf{s}, \alpha = 0, \beta = 1) \quad (42)$$

Therefore  $I_N^{d-GinSE}(L, \mathbf{t})$  is an example of the large ("fermionic") BKP tau function of [4] with respect to the variables  $N, L, \mathbf{t}$ . Then  $c(\mathbf{t}, \mathbf{s}) I_N^{dd-GinSE}(L, \mathbf{t}, \mathbf{s})$  and  $c(\mathbf{t}, \mathbf{s}) I_N^{dd-SE}(L, \mathbf{t}, \mathbf{s})$  are examples of the large 2-BKP tau function considered in [3] with respect to the variables  $N, L, \mathbf{t}, \mathbf{s}$  and of large 2-DKP tau function introduced in [20], see Appendix A.4.

Let us note that the presentation with  $\mathbf{s} = 0$ , namely, the representation for the symplectic Dyson-Wigner ensemble (where  $\alpha = 0$ ) was found by J. van de Leur in [5].

### 3.1 On perturbation series in deformation parameters

In [3] we have considered the following series over partitions

$$\sum_{\substack{\lambda \in \mathbb{P} \\ \ell(\lambda) \leq N}} \bar{A}_h(\lambda)(L) s_\lambda(\mathbf{t}) \quad (43)$$

where  $s_\lambda$  is the Schur function (see Appendix A.3) and  $\bar{A}_{h(\lambda)}(L)$  is a certain Pfaffian, see Appendix A.5. Such series one obtains for many kinds of matrix integrals, see [29], for instance for integrals over symplectic and orthogonal groups, see Appendix A.7. The notations are as follows.  $\lambda = (\lambda_1, \dots, \lambda_N)$ ,  $\lambda_1 \geq \dots \geq \lambda_N \geq 0$  is a partition, see [11] and  $h(\lambda) = \lambda_i - i + N$  are the so-called shifted parts of  $\lambda$ . The factors  $\bar{A}_h$  on the right-hand side of (43) are determined in terms a pair  $(A, a) =: \bar{A}$  where  $A$  is an infinite skew symmetric matrix and  $a$  an infinite vector. For a set  $h = (h_1, \dots, h_N)$ ,  $h_1 > \dots > h_N$  the numbers  $\bar{A}_h$  are defined as the Pfaffian of an antisymmetric  $2n \times 2n$  matrix  $\tilde{A}$  as follows:

$$\bar{A}_h(L) := \text{Pf}[\tilde{A}] \quad (44)$$

where for  $N = 2n$  even

$$\tilde{A}_{ij} = -\tilde{A}_{ji} := A_{h_i+L, h_j+L}, \quad 1 \leq i < j \leq 2n \quad (45)$$

and for  $N = 2n - 1$  odd

$$\tilde{A}_{ij} = -\tilde{A}_{ji} := \begin{cases} A_{h_i+L, h_j+L} & \text{if } 1 \leq i < j \leq 2n-1 \\ a_{h_i+L} & \text{if } 1 \leq i < j = 2n \end{cases}. \quad (46)$$

In addition we set  $\bar{A}_0 = 1$ .

Having the fermionic representation it is straightforward to write down the perturbation series of partition functions of Ginibre ensembles in the Schur functions like it was done in other cases (for instance see [23], [18], [17]).

(I) First let us do it for the quaternion real Ginibre ensemble. Re-writing

$$\int_{\mathbb{C}_+} (z - \bar{z}) \psi(z) \psi(\bar{z}) e^{-|z|^2} e^{-V(z^{-1}, \mathbf{s}) - V(\bar{z}^{-1}, \mathbf{s})} d^2 z = \sum_{n, m} A_{nm} \psi_n \psi_m \quad (47)$$

where  $A_{nm}$  is a moment matrix

$$A_{nm}^{GinSE}(\mathbf{s}) = \int_{\mathbb{C}_+} z^n \bar{z}^m (z - \bar{z}) e^{-|z|^2 - V(z^{-1}, \mathbf{s}) - V(\bar{z}^{-1}, \mathbf{s})} d^2 z, \quad a_n^{GinSE} = 0, \quad n, m = 1, \dots, N \quad (48)$$

we reduce problem to the solved in [3]. We obtain the following series in the Schur functions (see [11] for definitions and details)  $s_\lambda$  (please, do not mix with the deformation parameters  $s_i$ ):

$$I_N^{dd-GinSE}(L, \mathbf{t}, \mathbf{s}) = \sum_{\substack{\lambda \\ \ell(\lambda) \leq 2N}} \bar{A}_{\{h\}}^{GinSE}(L, \mathbf{s}) s_\lambda(\mathbf{t}) \quad (49)$$

In case  $\mathbf{s} = 0$  we get  $A_{m, n} = -A_{n, m} = n! \delta_{m+1, n}$ .

(II) For DW symplectic ensemble the formula is the same, but now instead of (48) we have

$$A_{nm}^{SE}(\mathbf{s}) = \frac{n-m}{2} \int_{\mathbb{R}} x^{n+m-1} e^{-x^2 - 2V(x^{-1}, \mathbf{s})} dx, \quad a_n^{SE} = 0, \quad n, m = 1, \dots, N \quad (50)$$

(III) For the DW orthogonal ensemble we have in case  $N$  is even the formula is the same, However for  $N$  odd we need to define  $A_{\{h\}}$ .

$$A_{nm}^{OE}(\mathbf{s}) = \int_{\mathbb{R}} \int_{\mathbb{R}} x^n y^m \text{sgn}(x - y) e^{-\frac{1}{2}x^2 - \frac{1}{2}y^2 - V(x^{-1}, \mathbf{s}) - V(y^{-1}, \mathbf{s})} dx dy, \quad (51)$$

$$a_n^{OE}(\mathbf{s}) = \int_{\mathbb{R}} x^n e^{-V(x^{-1}, \mathbf{s})} dx \quad (52)$$

(IV) For the real Ginibre ensemble we have

$$\begin{aligned} A_{nm}^{GinOE}(\mathbf{s}) &= \int_{\mathbb{C}_+} z^n \bar{z}^m \text{erfc}\left(\frac{|z - \bar{z}|}{\sqrt{2}}\right) e^{-\Re z^2 - V(z^{-1}, \mathbf{s}) - V(\bar{z}^{-1}, \mathbf{s})} d^2 z \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} x^n y^m \text{sgn}(x - y) e^{-\frac{1}{2}x^2 - \frac{1}{2}y^2 - V(x^{-1}, \mathbf{s}) - V(y^{-1}, \mathbf{s})} dx dy, \end{aligned} \quad (53)$$



$$a_n^{GinOE}(\mathbf{s}) = \int_{\mathbb{R}} x^n e^{-V(x^{-1}, \mathbf{s})} dx \quad (54)$$

For  $N$  even (which occurs in the cases of DW orthogonal and in real Ginibre ensembles) the formula for  $\bar{A}_{\{h\}}$  is slightly more spacious, see [3].

Let us also note that in all cases where  $\mathbf{s} = 0$  (namely, for d-OE, d-GinOE, d-SE, d-GinSE ensembles) moments can be explicitly evaluated in a straightforward way.

## 4 On Pfaffian formulae

Various Pfaffian formulae are known in the study of non-Hermitian matrices, see [26], [1]. These formulae, and perhaps some new ones, may be obtained by applying the Wick's rule to the evaluation of the fermionic expectation values, see Appendix A.5.

Example. Let us choose  $\mathbf{t}$  as  $\mathbf{t}$  shifted by the variables (9) where all  $p_i$  are different, all  $b_i = 0$ , all  $a_i = -1$  and  $N_1 = N$  even. Then according to relation (68) and the Wick's rule one can write the expectation value as a Pfaffian of the pairwise expectation values, see Appendix A.5. Then (10) is

$$\int \prod_i^N \det(1 - p_i X)^{-1} d\mu(X, L, \mathbf{t}, \mathbf{s}) = \frac{\prod_{i=1}^N p_i^{(L+1)(2-N)}}{\prod_{i>j} (p_i - p_j)} \text{Pf}[K]_{n,m=1,\dots,N} \quad (55)$$

where the fermionic representation for the tau function  $\tau_N(L, \mathbf{t}, \mathbf{s})$  (22) yields

$$K_{nm} = \langle L | \psi^\dagger(p_n) \psi^\dagger(p_m) e^\Phi | L \rangle = (p_m - p_n) K_{nm}^* \quad (56)$$

where  $K_{nm}^*$ :

$$K_{nm}^* = \int_{\mathbb{R}} x^L y^L \frac{|x - y| e^{-\frac{1}{2}x^2 - \frac{1}{2}y^2 + V(x, \mathbf{t}) + V(y, \mathbf{t}) - V(x^{-1}, \mathbf{s}) - V(y^{-1}, \mathbf{s})} dx dy}{(1 - xp_n)(1 - yp_n)(1 - xp_m)(1 - yp_m)} \quad \text{for OE} \quad (57)$$

$$K_{nm}^* = \int_{\mathbb{C}_+} (z - \bar{z}) |z|^{2L} \frac{\text{erfc}\left(\frac{|z - \bar{z}|}{\sqrt{2}}\right) e^{-\Re z^2 + V(z, \mathbf{t}) + V(\bar{z}, \mathbf{t}) - V(z^{-1}, \mathbf{s}) - V(\bar{z}^{-1}, \mathbf{s})} d^2 z}{(1 - zp_n)(1 - \bar{z}p_n)(1 - zp_m)(1 - \bar{z}p_m)} \quad (58)$$

$$+ \int_{\mathbb{R}} x^L y^L \frac{|x - y| e^{-\frac{1}{2}x^2 - \frac{1}{2}y^2 + V(x, \mathbf{t}) + V(y, \mathbf{t}) - V(x^{-1}, \mathbf{s}) - V(y^{-1}, \mathbf{s})} dx dy}{(1 - xp_n)(1 - yp_n)(1 - xp_m)(1 - yp_m)} \quad \text{for GinOE} \quad (59)$$

$$K_{nm}^* = \int_{\mathbb{R}} x^{2L} \frac{e^{-x^2 + 2V(x, \mathbf{t}) - 2V(x^{-1}, \mathbf{s})} dx}{(1 - xp_n)^2 (1 - xp_m)^2} \quad \text{for SE} \quad (60)$$

$$K_{nm}^* = \int_{\mathbb{C}_+} |z|^{2L} \frac{(z - \bar{z})^2 e^{-|z|^2 + V(z, \mathbf{t}) + V(\bar{z}, \mathbf{t}) - V(z^{-1}, \mathbf{s}) - V(\bar{z}^{-1}, \mathbf{s})} d^2 z}{(1 - zp_n)(1 - \bar{z}p_n)(1 - zp_m)(1 - \bar{z}p_m)} \quad \text{for GinSE} \quad (61)$$

Here  $\bar{z}$  is the complex conjugated to  $z$ . In this example the number of parameters  $p_i$  was equal to  $N$ . In other cases we typically obtain pfaffians of block matrices. This will be written down in a more detailed text.

## 5 Gauss complex Ginibre ensemble and its deformation

The complex Ginibre ensemble is considered to be easiest and relatively studied and it is quite similar to the well-known model of normal matrices [16], therefore I will skip details.

**Hermitian random matrices.** First we remind that the ensemble of random Hermitian matrices  $X$  with Gauss distribution for each entry was known as Gauss unitary ensemble (GUE). The deformation of the measure caused by multiplying the Gauss factor  $e^{-\text{tr} X}$  by  $e^{\text{tr} V(X, \mathbf{t})}$  was related to integrable systems in [12]. Let us add additional deformation parameters  $L, \mathbf{s}$ :

$$\int \det(X)^L e^{\sum_{m=1}^L ((t_m + t'_m) \text{tr} X^m - (s_m + s'_m) \text{tr} X^{-m}) - \text{tr} X^2} d\mu(X) \quad (62)$$

where  $d\mu(X)$  is the Haar measure on the space of Hermitian matrices (see [10] for details). Actually we do not need  $\mathbf{t}'$  and  $\mathbf{s}'$  but we keep it to compare the model with the non-Hermitian one. The fermionic representation for this model (with  $\mathbf{s} = \mathbf{s}' = 0$ ) was given in [13] (see also [15]). However to compare the result with the non-Hermitian case it is suitable to use another fermionic construction considered in [14] as follows

$$e_N \langle L + N, -N | \Gamma(\mathbf{t}, \mathbf{t}') e^{\int_{\mathbb{C}} \psi^{(1)}(x) \psi^{\dagger(2)}(\bar{x}) e^{-x^2} dx} \bar{\Gamma}(\mathbf{s}, \mathbf{s}') | L, 0 \rangle \quad (63)$$

with certain independent of deformation parameters factor  $e_N$ . Here we add the additional deformation caused by  $L, \mathbf{s}$ .

**Non-Hermitian random matrices** Complex restricted Ginibre ensemble with the double deformed measure

$$\int \det(X)^{L_1} \det(X^\dagger)^{-L_2} e^{\sum_{m=1} (t_m \text{tr} X^m + t'_m \text{tr} (X^\dagger)^m - s_m \text{tr} X^{-m} - s'_m \text{tr} (X^\dagger)^{-m}) - \text{tr} X X^\dagger} \prod_{i,j} dX_{ij} \quad (64)$$

(where parameters  $L_1, L_2, \mathbf{t}, \mathbf{t}', \mathbf{s}, \mathbf{s}'$  are chosen in a way that the integral is either convergent or may be regularized (see Remark 1)) is equal to the following two-component 2D Toda lattice [9] tau function

$$f_N \langle L_1 + N, L_2 - N | \Gamma(\mathbf{t}, \mathbf{t}') e^{\int_{\mathbb{C}} \psi^{(1)}(z) \psi^{\dagger(2)}(\bar{z}) e^{-|z|^2} d^2 z} \bar{\Gamma}(\mathbf{s}, \mathbf{s}') | L_1, L_2 \rangle \quad (65)$$

The factor  $f_N$  is independent of  $L_1, L_2, \mathbf{t}, \mathbf{t}', \mathbf{s}, \mathbf{s}'$ .

This type of representation was previously used in [17] in the context of two-matrix and normal matrix models. Perturbation series in deformation parameters for the complex Ginibre ensemble are basically the same as found in [17].

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## A Appendices

### A.1 Vertex operators

Vertex operators we need are as follows

$$\hat{X}(L, \mathbf{t}, \lambda) := e^{\sum_{n=1}^{\infty} \lambda^n t_n} \lambda^L e^{-\sum_{n=1}^{\infty} \frac{1}{n\lambda^n} \frac{\partial}{\partial t_n}}, \quad \hat{X}^\dagger(L, \mathbf{t}, \lambda) := e^{-\sum_{n=1}^{\infty} \lambda^n t_n} \lambda^{-L} e^{\sum_{n=1}^{\infty} \frac{1}{n\lambda^n} \frac{\partial}{\partial t_n}} \quad (66)$$

$$\hat{Y}(L, \mathbf{s}, \lambda) := e^{-\sum_{n=1}^{\infty} \lambda^{-n} s_n} \lambda^L e^{\sum_{n=1}^{\infty} \frac{\lambda^n}{n} \frac{\partial}{\partial s_n}}, \quad \hat{Y}^\dagger(L, \mathbf{s}, \lambda) := e^{\sum_{n=1}^{\infty} \lambda^{-n} s_n} \lambda^{-L} e^{-\sum_{n=1}^{\infty} \frac{\lambda^n}{n} \frac{\partial}{\partial s_n}} \quad (67)$$

Interesting historical fact that the formula which relates fermions to bosons first was found in [22].

The following bosonization relation is useful

$$\langle L + N | \Gamma_+(\mathbf{t} + \sum_{i=1}^N [p_i]) \rangle = \frac{\langle L | \psi^\dagger(p_1^{-1}) \cdots \psi^\dagger(p_N^{-1}) \Gamma_+(\mathbf{t}) \rangle}{\prod_{i=1}^N p_i^{(L+1)(N-1)} \prod_{i>j} (p_i - p_j)} \quad (68)$$

Introduce

$$\hat{\Omega}_n(L, \mathbf{t}) = \operatorname{res}_\lambda \left( \frac{\partial^n \hat{X}(L, \mathbf{t}, \lambda)}{\partial \lambda^n} \hat{X}^\dagger(L, \mathbf{t}, \lambda) \right), \quad \hat{\Omega}_n^*(L, \mathbf{s}) = \operatorname{res}_\lambda \left( \hat{Y}^\dagger(L, \mathbf{s}, \lambda) \frac{\partial^n \hat{Y}(L, \mathbf{s}, \lambda)}{\partial \lambda^n} \right) \quad (69)$$

### A.2 Fermions

We shall remind some facts and notations of [9]. Introduce free fermionic fields  $\psi(z) = \sum_{i \in \mathbb{Z}} \psi_i z^i$ ,  $\psi^\dagger(z) = \sum_{i \in \mathbb{Z}} \psi_{-i-1}^\dagger z^i$  whose Fourier components anti-commute as follows  $\psi_i \psi_j + \psi_j \psi_i = \psi_i^\dagger \psi_j^\dagger + \psi_j^\dagger \psi_i^\dagger = 0$  and  $\psi_i \psi_j^\dagger + \psi_j^\dagger \psi_i = \delta_{i,j}$  where  $\delta_{i,j}$  is the Kronecker symbol. We put

$$\psi_i |0\rangle = \psi_{-i}^\dagger |0\rangle = \langle 0 | \psi_{-i} = \langle 0 | \psi_i^\dagger = 0 \quad (70)$$

where  $\langle 0 |$  and  $|0\rangle$  are left and right vacuum vectors of the fermionic Fock space,  $\langle 0 | \cdot 1 \cdot |0\rangle = 1$ . Also introduce

$$\langle n | = \begin{cases} \langle 0 | \psi_0^\dagger \cdots \psi_{n-1}^\dagger & \text{if } n > 0 \\ \langle 0 | \psi_{-1} \cdots \psi_{-n} & \text{if } n < 0 \end{cases}, \quad |n\rangle = \begin{cases} \psi_{n-1} \cdots \psi_0 |0\rangle & \text{if } n > 0 \\ \psi_{-n}^\dagger \cdots \psi_{-1}^\dagger |0\rangle & \text{if } n < 0 \end{cases} \quad (71)$$

Then  $\langle n | \cdot 1 \cdot |m\rangle = \delta_{n,m}$ .

Following [4] we introduce an additional Fermi mode which we shall denote by  $\phi$  with properties <sup>3</sup>

$$\phi \psi_i + \psi_i \phi = \phi \psi_i^\dagger + \psi_i^\dagger \phi = 0, \quad \phi^2 = \frac{1}{2} \quad (72)$$

$$\phi |0\rangle = |0\rangle \frac{1}{\sqrt{2}}, \quad \langle 0 | \phi = \frac{1}{\sqrt{2}} \langle 0 | \quad (73)$$

such that  $\langle L | \phi | L \rangle = \frac{(-)^L}{\sqrt{2}}$ .

Now, two-component Fermi fields used in Section 5 are defined as

$$\psi^{(i)}(z) = \sum_{n \in \mathbb{Z}} z^n \psi_{2n+i}, \quad \psi^{(i)\dagger}(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} \psi_{2n+1}^\dagger \quad (74)$$

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<sup>3</sup>In notations of [4] our  $\psi_n$ ,  $\psi_n^\dagger$  and  $\phi$  read respectively as  $\psi_{n+\frac{1}{2}}$ ,  $\psi_{n+\frac{1}{2}}^\dagger$  and  $\psi_0$

Other details about two-component fermions may be found in [9] or in [17].

We have

$$\hat{X}(L, \mathbf{t}, \lambda) X^\dagger(L, \mu) \langle N + L | \Gamma_+(\mathbf{t}) g \Gamma_-(\mathbf{s}) | L \rangle = \langle N + L | \Gamma_+(\mathbf{t}) \psi(\lambda) \psi^\dagger(\mu) g \Gamma_-(\mathbf{s}) | L \rangle \quad (75)$$

$$\hat{Y}^\dagger(L, \mathbf{s}, \mu) Y(\lambda) \langle N + L | \Gamma_+(\mathbf{t}) g \Gamma_-(\mathbf{s}) | L \rangle = \langle N + L | \Gamma_+(\mathbf{t}) g \psi(\lambda) \psi^\dagger(\mu) \Gamma_-(\mathbf{s}) | L \rangle \quad (76)$$

Then it follows that

$$\hat{\Omega}_n(L, \mathbf{t}) \langle N + L | \Gamma_+(\mathbf{t}) g \Gamma_-(\mathbf{s}) | L \rangle = \langle N + L | \Gamma_+(\mathbf{t}) \tilde{\Omega}_n g \Gamma_-(\mathbf{s}) | L \rangle \quad (77)$$

$$\tilde{\Omega}_n(L, \mathbf{t}) \langle N + L | \Gamma_+(\mathbf{t}) g \Gamma_-(\mathbf{s}) | L \rangle = \langle N + L | \Gamma_+(\mathbf{t}) g \tilde{\Omega}_n \Gamma_-(\mathbf{s}) | L \rangle \quad (78)$$

where

$$\tilde{\Omega}_n = \text{res}_\lambda \left( \frac{\partial^n \psi(\lambda)}{\partial \lambda^n} \psi^\dagger(\lambda) \right) \quad (79)$$

Using the fermionic representation one may verify that tau functions related to the deformed orthogonal and deformed symplectic ensembles obey the constraints

$$\left( \hat{\Omega}_n(L, \mathbf{t}) - \hat{\Omega}_n^*(L, \mathbf{s}) \right) \tau(L, \mathbf{t}', \mathbf{s}') = 0, \quad n \geq 1 \quad \text{odd} \quad (80)$$

where  $t'_k = t_k - \frac{1}{2} \delta_{2,k}$ ,  $s'_k = s_k - \frac{1}{2} \delta_{2,k}$  (this shift appears due to the Gauss measure in undeformed ensembles).

### A.3 The Schur function

Consider polynomials  $h_n(\mathbf{t})$  defined by  $e^{\sum_{n=1}^\infty z^n t_n} = \sum_{n=0}^\infty z^n h_n(\mathbf{t})$ . Then the Schur function labeled by a partition  $\lambda = (\lambda_1, \dots, \lambda_k > 0)$  may be defined as  $s_\lambda(\mathbf{t}) = \det(h_{\lambda_i - i + j}(\mathbf{t}))_{i,j=1,\dots,k}$ .

### A.4 From 2-BKP to TL, BKP, 2-DKP, DKP

The general expression for  $\Phi$  of (22) is as follows

$$\Phi = \sum_{i,j \in \mathbb{Z}} A_{ij} \psi_i \psi_j + \sum_{i,j \in \mathbb{Z}} B_{ij} \psi_i^\dagger \psi_j^\dagger + \sum_{i,j \in \mathbb{Z}} D_{ij} \psi_i \psi_j^\dagger + \phi \sum_{i \in \mathbb{Z}} a_i \psi_i + \phi \sum_{i \in \mathbb{Z}} b_i \psi_i^\dagger \quad (81)$$

To get TL tau function [9], [19] we put all  $A_{ij}, B_{ij}, a_i, b_i$  and  $N$  to be zero.

To get BKP [4] we put  $\mathbf{s} = 0$ .

To get 2-DKP [20] we put all  $a_i = b_i = 0$ .

To get DKP [9] we put  $\mathbf{s} = 0$  and all  $a_i = b_i = 0$ .

Tau functions (23) and (38) correspond to the case where all  $B_{ij}, D_{ij}, b_i$  vanish (and for (38) also all  $a_i = 0$ ). The case where only  $A_{ij}$  (and perhaps  $a_i$ ) are non-vanishing is characterized by the condition that the so-called wave functions

$$w^{(\infty)}(N, L, \mathbf{t}, \mathbf{s}, \lambda) = \frac{\hat{X}(L, \mathbf{t}, \lambda) \tau_N(L, \mathbf{t}, \mathbf{s})}{\tau_N(L, \mathbf{t}, \mathbf{s})} = \lambda^L e^{V(\lambda, \mathbf{t}) - V(\lambda^{-1}, \mathbf{s})} P_N(L, \mathbf{t}, \mathbf{s}, \lambda) \quad (82)$$

$$w^{(0)}(N, L, \mathbf{t}, \mathbf{s}, \lambda) = \frac{\hat{Y}^\dagger(L, \mathbf{s}, \lambda) \tau_N(L, \mathbf{t}, \mathbf{s})}{\tau_N(L, \mathbf{t}, \mathbf{s})} = \lambda^L e^{V(\lambda, \mathbf{t}) - V(\lambda^{-1}, \mathbf{s})} Q_N(L, \mathbf{t}, \mathbf{s}, \lambda) \quad (83)$$

are equal, and  $P_N = Q_N$  are polynomials in  $\lambda$  of the order  $N$ .

### A.5 Pfaffians

If  $A$  an anti-symmetric matrix of an odd order its determinant vanishes. For even order, say  $k$ , the following multilinear form in  $A_{ij}, i < j \leq k$

$$\text{Pf}[A] := \sum_{\sigma} \text{sgn}(\sigma) A_{\sigma(1), \sigma(2)} A_{\sigma(3), \sigma(4)} \cdots A_{\sigma(k-1), \sigma(k)} \quad (84)$$

where sum runs over all permutation restricted by

$$\sigma : \sigma(2i-1) < \sigma(2i), \quad \sigma(1) < \sigma(3) < \cdots < \sigma(k-1), \quad (85)$$

coincides with the square root of  $\det A$  and is called the *Pfaffian* of  $A$ , see, for instance [10]. As one can see the Pfaffian contains  $1 \cdot 3 \cdot 5 \cdots (k-1) = (k-1)!!$  terms.

**Wick's relations.** Let each of  $w_i$  be a linear combination of Fermi operators:

$$\hat{w}_i = \sum_{m \in \mathbb{Z}} v_{im} \psi_m + \sum_{m \in \mathbb{Z}} u_{im} \psi_m^\dagger, \quad i = 1, \dots, n$$

Then the Wick formula is

$$\langle l | \hat{w}_1 \cdots \hat{w}_n | l \rangle = \begin{cases} \text{Pf}[A]_{i,j=1,\dots,n} & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases} \quad (86)$$

where  $A$  is  $n$  by  $n$  antisymmetric matrix with entries  $A_{ij} = \langle l | \hat{w}_i \hat{w}_j | l \rangle$ ,  $i < j$ .

## A.6 Hirota equations

Hirota equations for the large BKP hierarchy were written in [4]. For 2-BKP hierarchy Hirota equations are as follows [3]

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{N'+l'-N-l-2} e^{V(\mathbf{t}'-\mathbf{t},z)} \tau_{N'-1}(l', \mathbf{t}' - [z^{-1}], \mathbf{s}') \tau_{N+1}(l, \mathbf{t} + [z^{-1}], \mathbf{s}) \\ & + \oint \frac{dz}{2\pi i} z^{N+l-N'-l'-2} e^{V(\mathbf{t}-\mathbf{t}',z)} \tau_{N'+1}(l', \mathbf{t}' + [z^{-1}], \mathbf{s}') \tau_{N-1}(l, \mathbf{t} - [z^{-1}], \mathbf{s}) \\ & = \oint \frac{dz}{2\pi i} z^{l'-l} e^{V(\mathbf{s}'-\mathbf{s},z^{-1})} \tau_{N'-1}(l' + 1, \mathbf{t}', \mathbf{s}' - [z]) \tau_{N+1}(l - 1, \mathbf{t}, \mathbf{s} - [z]) \\ & + \oint \frac{dz}{2\pi i} z^{l-l'} e^{V(\mathbf{s}'-\mathbf{s},z^{-1})} \tau_{N'+1}(l' - 1, \mathbf{t}', \mathbf{s}' + [z]) \tau_{N-1}(l + 1, \mathbf{t}, \mathbf{s} + [z]) \\ & + \frac{(-1)^{l'+l}}{2} (1 - (-1)^{N'+N}) \tau_{N'}(l', \mathbf{t}', \mathbf{s}') \tau_N(l, \mathbf{t}, \mathbf{s}) \end{aligned} \quad (87)$$

The difference Hirota equation may be obtained from the previous one [3]

$$\begin{aligned} & -\frac{\beta}{\alpha - \beta} \tau_N(l, \mathbf{t} + [\beta^{-1}]) \tau_{N+1}(l, \mathbf{t} + [\alpha^{-1}]) - \frac{\alpha}{\beta - \alpha} \tau_N(l, \mathbf{t} + [\alpha^{-1}]) \tau_{N+1}(l, \mathbf{t} + [\beta^{-1}]) \\ & + \frac{1}{\alpha\beta} \tau_{N+2}(l, \mathbf{t} + [\alpha^{-1}] + [\beta^{-1}]) \tau_{N-1}(l, \mathbf{t}) = \tau_{N+1}(l, \mathbf{t} + [\alpha^{-1}] + [\beta^{-1}]) \tau_N(l, \mathbf{t}). \end{aligned} \quad (88)$$

## A.7 Integrals over orthogonal and symplectic groups

Using

$$\int_{O \in \mathbb{O}(N)} s_\lambda(O) d_* O = \begin{cases} 1 & \lambda \text{ is even} \\ 0 & \text{otherwise} \end{cases}, \quad \int_{S \in \mathbb{Sp}(N)} s_\lambda(S) d_* S = \begin{cases} 1 & \lambda^{tr} \text{ is even} \\ 0 & \text{otherwise} \end{cases} \quad (89)$$

we get

$$J_1(\mathbf{t}, N) := \int_{O \in \mathbb{O}(N)} e^{\sum_{m=1}^\infty t_m \text{Tr} O^m} d_* O = \sum_{\substack{\lambda \text{ even} \\ \ell(\lambda) \leq N}} s_\lambda(\mathbf{t}) \quad (90)$$

$$J_2(\mathbf{t}, N) := \int_{S \in \mathbb{Sp}(2n)} e^{\sum_{m=1}^\infty t_m \text{Tr} S^m} d_* S = \sum_{\substack{\lambda^{tr} \text{ even} \\ \ell(\lambda) \leq 2n}} s_\lambda(\mathbf{t}) \quad (91)$$

The right hand sides were obtained in [3] as examples of the BKP tau function. Thus integrals  $J_1(\mathbf{t}, N)$  and  $J_2(\mathbf{t}, N)$  are tau functions,  $N$  and  $\mathbf{t}$  being BKP higher times.